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Stiffness Design Method of Symmetric Laminates Using Lamination Parameters

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Introduction

IN the stiffness design of laminated composites, it is important to optimize layer angles as well as layer thicknesses. Introducing lamination parameters as design variables is efficient and reliable in the optimization of laminated composites since the stiffness components of laminated composites are expressed as a linear function with respect to lamination parameters.[1] To use lamination parameters in the design of laminated composites, the feasible region of lamination parameters needs to be known. The method for determining the laminate configurations corresponding to the lamination parameters also has to be established. The previous paper[2] has shown those fundamental relations for a specially orthotropic laminate eliminating coupling terms.

In the symmetric laminate with the extension-shear coupling or the bending-twisting coupling, the in-plane and out-of-plane stiffness characteristics are governed by four in-plane and four out-of-plane lamination parameters, respectively. The present paper shows the feasible region of in-plane or out-of-plane lamination parameters for the symmetric laminate. A method is proposed for determining laminate configurations corresponding to the lamination parameters.

Stiffness Characteristics of Symmetric Laminates

In the classical lamination theory, the constitutive equation of symmetric laminates is given by

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix} \tag{1}$$

where N and M denote the stress and moment resultants, respectively; ϵ and κ denote the strains and the curvature changes at the midplane, respectively; and A_{ij} and D_{ij} represent the in-plane and the out-of-plane stiffnesses, respectively.

When the stiffness components A_{ij} and D_{ij} $(i,j = 1,2,6)$ are expressed by the stiffness invariants and the lamination parameters,[1] A_{ij} and D_{ij} are governed by four in-plane and four out-of-plane lamination parameters, respectively,

$$\begin{aligned} \xi_1 &= \int_0^1 \cos 2\theta(u)\,\mathrm{d}u, \quad \xi_2 = \int_0^1 \cos 4\theta(u)\,\mathrm{d}u \\ \xi_3 &= \int_0^1 \sin 2\theta(u)\,\mathrm{d}u, \quad \xi_4 = \int_0^1 \sin 4\theta(u)\,\mathrm{d}u \end{aligned} \tag{2}$$

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$$\begin{aligned} \xi_9 &= 3\int_0^1 \cos 2\theta(u)u^2\,\mathrm{d}u, \quad \xi_{10} = 3\int_0^1 \cos 4\theta(u)u^2\,\mathrm{d}u \\ \xi_{11} &= 3\int_0^1 \sin 2\theta(u)u^2\,\mathrm{d}u, \quad \xi_{12} = 3\int_0^1 \sin 4\theta(u)u^2\,\mathrm{d}u \end{aligned} \tag{3}$$

where $\theta(u)$ is a distribution function of the fiber angles through the thickness, and $(\xi_1, \xi_2, \xi_3, \xi_4)$ and $(\xi_9, \xi_{10}, \xi_{11}, \xi_{12})$ represent the in-plane and out-of-plane lamination parameters, respectively. In Eq. (1), the stiffness components are a linear function with respect to the lamination parameters. In the stiffness optimization, the use of lamination parameters as design variables leads to an efficient and reliable optimization approach.

The lamination parameters depend on each other. The relation between the lamination parameters has not been known, although the preceding study was performed in Refs. 3 and 4. The present paper first examines the relation between the four in-plane lamination parameters. As shown in Fig. 1, the feasible region of a point $Q(\xi_2,\xi_4)$ for a fixed point $P(\xi_1, \xi_3)$ is expressed as follows:

$$(\xi_2 - r^2 \cos 2\alpha)^2 + (\xi_4 - r^2 \sin 2\alpha)^2 \le (1 - r^2)^2 \tag{4}$$

where $r \cos \alpha = \xi_1$ and $r \sin \alpha = \xi_3$. Equation (4) is also expressed as follows:

$$\left| \int_0^1 \exp(i4\theta)\,\mathrm{d}u - \left[\int_0^1 \exp(i2\theta)\,\mathrm{d}u \right]^2 \right| \le 1 - \left| \int_0^1 \exp(i2\theta)\,\mathrm{d}u \right|^2 \tag{5}$$

where $|z|$ denotes the absolute value of a complex number z. We can prove the relation of Eq. (5) easily.

When the coupling terms vanish, i.e., $\xi_3 = \xi_4 = 0$, Eq. (4) leads to the following relation:

$$2\xi_1^2 - 1 \le \xi_2 \le 1 \tag{6}$$

Equation (6) gives the feasible region of lamination parameters for the laminate without extension-shear coupling.

Equation (4) shows the feasible region of (ξ_2, ξ_4) for the fixed values of (ξ_1, ξ_3). On the other hand, for the fixed values of (ξ_1, ξ_2), Eq. (4) is transformed as follows:

$$2(1+\xi_2)\xi_3^2 - 4\xi_1\xi_3\xi_4 + \xi_4^2 \le (\xi_2 - 2\xi_1^2 + 1)(1 - \xi_2) \tag{7}$$

Equation (7) shows that the feasible region of (ξ_3, ξ_4) is within an ellipse for $1+\xi_2-2\xi_1^2>0$ and on the straight line of $\xi_4 = 2\xi_1\xi_3$ $(-1 \le \xi_1 \le 1)$ for $1+\xi_2-2\xi_1^2=0$. When the lamination parameters $((\xi_1, \xi_2)$ are specified, the in-plane stiffness components, A_{11}, A_{12}, A_{22}, and A_{66}, are determined uniquely. On the other hand, the extension-shear coupling terms, A_{16}

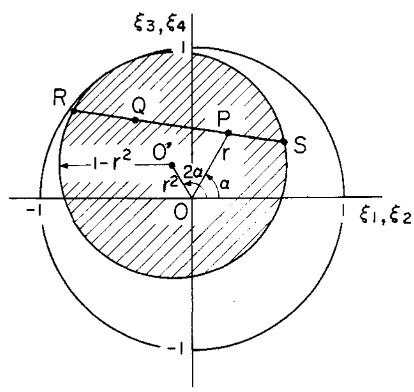
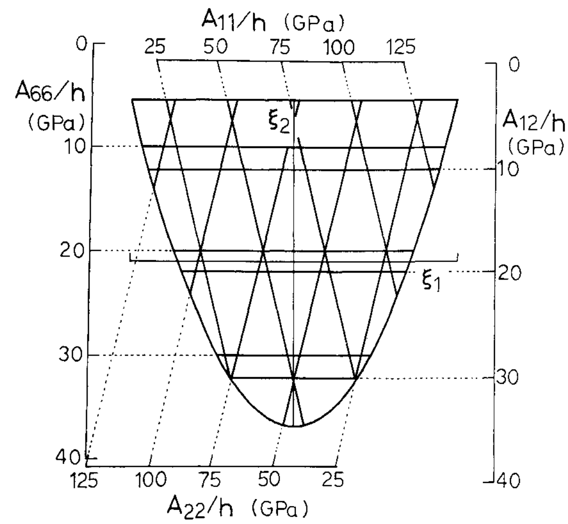
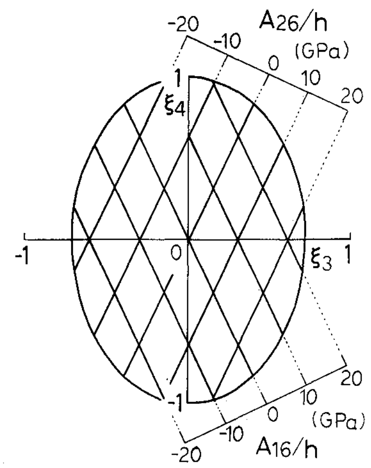


Fig. 1 Feasible region of $Q(\xi_2, \xi_4)$ for $P(\xi_1, \xi_3)$.



a) A_{11}, A_{12}, A_{22}, and A_{66}



b) A_{16} and A_{26}

Fig. 2 Contours of stiffness components on the lamination parameter plane.

and A_{26}, are determined by specifying the coupling lamination parameters (ξ_3,ξ_4). Figure 2a shows the contours of the in-plane stiffness components, A_{11}/h, A_{12}/h, A_{22}/h, and A_{66}/h, on the lamination parameter plane $\xi_1 - \xi_2$ where h is the thickness of the plate and the elastic properties of graphite /epoxy composites are used ($E_L = 142$ GPa, $E_T = 10.8$ GPa, $G_{LT} = 5.49$ GPa, $\nu_L = 0.3$). Figure 2b shows the contours of the in-plane coupling stiffness components A_{16}/h and A_{26}/h on the $\xi_3 - \xi_4$ plane, where the feasible region of $2\xi_3^2 + \xi_4^2 \leq 1$ corresponds to the case of (ξ_1, ξ_2) = (0, 0). Thus we can easily determine a set of lamination parameters (ξ_1, ξ_2, ξ_3, ξ_4) corresponding to the required in-plane stiffnesses.

We have derived the relation between four in-plane lamination parameters. The relation between the out-of-plane lamination parameters is obtained in the same manner when replacing (ξ_1, ξ_2, ξ_3, ξ_4) with (ξ_9, ξ_{10}, ξ_{11}, ξ_{12},) in Eqs. (4), (6), and (7).

Next we consider the feasible region of lamination parameters when the layer angles are preassigned. Figure 3 shows the feasible regions of (ξ_3, ξ_4) for (ξ_1, ξ_2) = (0, 0) in $0/\pm 45/90$ laminates and $0/\pm\theta/90$ laminates, respectively, where the layer thicknesses and the layer angle θ are taken as variables. The maximum feasible region is $2\xi_3^2 + \xi_4^2 \leq 1$ when (ξ_1, ξ_2) = (0, 0). The feasible region for $0/\pm 45/90$ laminates is on the straight line of $\xi_4 = 0$ ($-1/2 \leq \xi_3 \leq 1/2$) whereas that for $0/\pm\theta/90$ laminates is expressed by $4\xi_3^2 - \xi_4^2 \leq 1$ and $|\xi_4| \leq |\xi_3|$. The feasible region for $0/\pm\theta/90$ laminates is greater than that for $0/\pm 45/90$ laminates, but it is still restricted to a small region as compared with the maximum feasible region. In the aeroelastic tailoring of the aircraft composite wing, the utilization of coupling effects is essential, and thus more degrees of freedom in the stiffness design are necessary for tailored design than that of $0/\pm 45/90$ or $0/\pm\theta/90$ laminates.

Method for Determining Laminate Configurations

We consider a method for determining laminate configurations corresponding to the in-plane lamination parameters within the feasible region. The fundamental equation is given as follows:

$$\sum_{j=1}^{J} h_j \exp(i2\theta_j) = \xi_1 + i\xi_3$$
$$\sum_{j=1}^{J} h_j \exp(i4\theta_j) = \xi_2 + i\xi_4 \tag{8}$$

The problem is to obtain h_j and θ_j for the given values of (ξ_1, ξ_2, ξ_3, ξ_4). The general solution method has not been established; however, we can obtain the solution of Eq. (8) for the case of $J = 4$ from geometrical considerations.

In Fig. 1, two points are denoted by R and S where the points are the intersection of the straight line PQ and the small circle representing the equality relation of Eq. (4). Then the laminate configuration for $J = 4$ is expressed using the coordinates of the points R and S as follows:

$$[(\theta_1)_{h_1}/(\theta_2)_{h_2}/(\theta_3)_{h_3}/(\theta_4)_{h_4}]_S \text{ laminate}$$
$$\theta_{1,2} = (\pi + \beta_R \mp 2_{\gamma_R})/4 \tag{9}$$
$$\theta_{3,4} = (\pi + \beta_S \mp 2_{\gamma_S})/4$$

where

$$\cos\beta_R = \frac{x_R - r^2\cos 2\alpha}{1 - r^2}, \quad \sin\beta_R = \frac{y_R - r^2\sin 2\alpha}{1 - r^2}$$
$$\cos\gamma_R = -r\sin\frac{\beta_R - 2\alpha}{2}, \quad \cos\beta_S = \frac{x_S - r^2\cos 2\alpha}{1 - r^2} \tag{10}$$
$$\sin\beta_S = \frac{y_S - r^2\sin 2\alpha}{1 - r^2}, \quad \cos\gamma_S = -r\sin\frac{\beta_S - 2\alpha}{2}$$
$$r\cos\alpha = \xi_1, \quad r\sin\alpha = \xi_3$$

and

$$h_1 = \frac{PA_2}{A_1A_2}\frac{QS}{RS}, \quad h_2 = \frac{PA_1}{A_1A_2}\frac{QS}{RS}$$
$$h_3 = \frac{PA_4}{A_3A_4}\frac{RQ}{RS}, \quad h_4 = \frac{PA_3}{A_3A_4}\frac{RQ}{RS} \tag{11}$$

where $A_i = (\cos 2\theta_i, \sin 2\theta_i)$ ($i = 1, 2, 3, 4$).

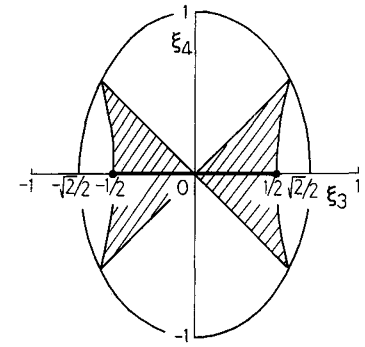


Fig. 3 Feasible region of (ξ_3, ξ_4) for (ξ_1, ξ_2) = (0, 0) in conventional laminates.

Table 1 Laminate configurations of $[(\theta_1)_{h_1}/(\theta_2)_{h_2}/(\theta_3)_{h_3}/(\theta_4)_{h_4}]_s$ corresponding to in-plane lamination parameters

$(\xi_1, \xi_2, \xi_3, \xi_4)$	θ_1, deg	θ_2, deg	θ_3, deg	θ_4, deg	h_1	h_2	h_3	h_4
(0.1, −0.2, 0.4, 0.6)	32.8	−64.3	10.8	77.0	0.51	0.22	0.15	0.12
(0.5, 0.5, 0, −0.2)	−9.2	64.0	9.2	−64.0	0.46	0.18	0.26	0.10
(0.3, 0.4, 0.4, 0)	22.2	−76.2	3.5	64.6	0.17	0.06	0.44	0.33

Table 1 shows examples of laminate configurations for some sets of $(\xi_1, \xi_2, \xi_3, \xi_4)$. The method for determining the laminate configurations corresponding to the out-of-plane lamination parameters $(\xi_9, \xi_{10}, \xi_{11}, \xi_{12})$ is similar. The layer angles are obtained using Eq. (9). The layer thicknesses of the $[(\theta_1)_{\bar{h}_1}/(\theta_2)_{\bar{h}_2}/(\theta_3)_{\bar{h}_3}/(\theta_4)_{\bar{h}_4}]_s$ laminate are determined from the following relations:

$$\bar{h}_j = \sqrt[3]{h_j + h_{j+1} + \cdots + h_4} - \sqrt[3]{h_{j+1} + \cdots + h_4}, \quad (j = 1, 2, 3, 4) \tag{12}$$

where h_j is the thickness component shown in Eq. (11).

Conclusions

The present paper has shown a stiffness design method of symmetric laminates using lamination parameters. The relation between the lamination parameters in symmetric laminates has been obtained. The method has been shown for determining the laminate configurations corresponding to the in-plane lamination parameters $(\xi_1, \xi_2, \xi_3, \xi_4)$ or out-of-plane lamination parameters $(\xi_9, \xi_{10}, \xi_{11}, \xi_{12})$. The limitation of the conventional laminates has also been discussed from the design viewpoint.

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Vibration of Compressively Loaded Shear Deformable Flat Panels Exhibiting Initial Geometric Imperfections

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Introduction

ONE of the features characterizing the design of next-generation aeronautical/aerospace vehicles consists of the incorporation of advanced composite materials in their construction. For a more rational design of these structures as well as for a more exhaustive utilization of their unique properties, a better knowledge of their response characteristics is needed.

As is well known, composite material structures, in contrast to their metallic counterparts, exhibit high flexibilities in transverse shear, which, in many instances, result in detrimental effects. In the case of an orthotropic body, the transverse flexibilities are measured in terms of the ratios $E_\alpha/G_{\alpha 3}$ $(\alpha = 1, 2)$, whereas in the case of a transversely isotropic material, they are measured in terms of the ratio E/G', where $E_\alpha(E)$ and $G_{\alpha 3}(G')$ denote the in-plane Young's and transverse shear moduli, respectively. One of the goals of this Note is to assess the influence played by the transverse shear flexibility effect on small-amplitude vibrations of homogeneous/composite laminated panels compressed by edge loads in the pre- and postbuckling ranges. Because of the importance played by the unavoidable geometric imperfections and by the character of in-plane boundary conditions, their influence on the considered problem will also be investigated.

In spite of its evident importance, the study of the vibrational behavior of preloaded imperfect composite panels was done, with the exception of Ref. 1 within the classical Kirchhoff's model. It could easily be inferred, however, that employment of this model, postulating an infinite rigidity in transverse shear, instead of a more refined one incorporating this effect, will result, in many instances, in erroneous predictions of the response characteristics. In connection with the problem considered herein, the illustration of this statement will be given later in the paper.

Basic Assumptions: Governing Equations

The case of a flat plate of uniform thickness h symmetrically laminated of $2m + 1 (m = 1, 2, \ldots)$ transversely isotropic elastic layers is considered. It is assumed that the planes of isotropy of each material layer are parallel at each point to the reference plane of the composite panel (selected to coincide with the midplane of the midlayer).

It is assumed that the layers are in perfect bond, implying that no slip between two contiguous layers may occur.

The points of the three-dimensional space of the plate are referred to a set of rectangular Cartesian normal system of coordinates x_i, where x_α $(\alpha = 1, 2)$ denotes the in-plane coordinates, whereas x_3 denotes the normal one to the plane $x_3 = 0$ (defining the reference plane). Extension to the dynamic case of the geometrically nonlinear higher-order shear-deformable theory (HSDT) of laminated plates developed for the static case in Ref. 2 results in the following governing equations:

$$D \nabla^4 u_3 - c_{\alpha\omega} c_{\beta\rho} \{ [u_{3,\alpha\beta} + \mathring{u}_{3,\alpha\beta}] F_{,\omega\rho} - \Omega \nabla^2 [F_{,\omega\rho}(u_{3,\alpha\beta} + \mathring{u}_{3,\alpha\beta})] \}$$

$$+ m_0 \left\{ \ddot{u}_3 - \left(\frac{B + C}{S} + \delta_A \delta_B \left[\frac{R}{m_0} - \frac{M}{S} \right] \right) \nabla^2 \ddot{u}_3 \right\} = 0 \tag{1a}$$